Free Vibration of Stiffened Rectangular Plates Using Green's Functions and Integral Equations

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A new method for the free-vibration analysis of stiffened rectangular plates based on the use of Green's functions and the solution of a system of Fredholm integral equations of the second kind is demonstrated. The lateral forces and twisting moments of constraint between the plate and beam stiffeners are accounted for. For plates with simply supported edges perpendicular to the stiffeners the integral equations are solved exactly to yield the characteristic equations for the natural frequencies. The characteristic equations are then solved to yield the exact natural frequencies (within the context of the mathematical model chosen). The pertinent parameters of the problem all appear explicitly in the characteristic equation which then has implications for its use as a mathematical design tool. The exact natural modes can be calculated and the orthogonality relation for the natural modes is given which implies that the forced response of the stiffened plate, including the effects of damping, can be determined by modal analysis. As an example, a table and figures show the natural frequencies for a stiffened simply supported rectangular plate with a single stiffener.

Introduction

STIFFENED plates occur in many aeronautical, aerospace, and marine structures. They can also be found in the automotive industry and as bridge decks. The free-vibration analysis to determine their natural frequencies has been performed in a number of ways: 1) by considering a structurally equivalent orthotropic plate,[1,2] 2) by considering the classical theories for an isotropic plate and beam stiffeners and matching their individual solutions (sometimes called the composite beam-plate method),[3,4] 3) by various matrix methods,[5-7] 4) the Rayleigh-Ritz method,[8] 5) Lagrange multiplier methods,[9,10] 6) the modal constraint method,[11] 7) the finite difference method in conjunction with a variational principle,[12] and 8) the finite element method.[13] In addition, the receptance method has been applied in the free-vibration analysis of ring-stiffened cylindrical shells.[14,15] (For a more complete bibliography of the literature on vibrating stiffened plates one should also consult the references of the above-mentioned papers.) The orthotropic plate provides an approximate physical model for the stiffened plate, while requiring static experimental testing to determine the necessary orthotropic elastic constants. The composite beam-plate method and the matrix methods provide exact natural frequencies within the mathematical model of the stiffened plate but involve many unknowns and are not readily extended to analysis of the forced response of the stiffened plate. The receptance method also provides exact natural frequencies but is not readily extended to the forced-response analysis. The remaining methods all involve approximate solution techniques.

In this paper a new method for the free-vibration analysis of stiffened plates, based on the use of Green's functions and the solution of a system of homogeneous Fredholm integral equations of the second kind, will be demonstrated. The natural frequencies and natural modes are exact within the mathematical model chosen for the stiffened plate, and the results are readily used in the forced-response analysis. The use of Green's functions in the vibration analysis of linear combined dynamical systems involving plates and oscillators has been demonstrated previously by the author.[16] The static deflection analysis of a simply supported stiffened rectangular plate using Green's functions has been studied previously by Michalopoulos and Wheeler.[17] The determination of natural frequencies and natural modes of stiffened plates through the derivation and solution of integral equations, to the author's knowledge, has not previously appeared.

For the present analysis, the stiffeners are all in one direction and parallel to the edges of the rectangular plate. The plate is of uniform thickness and simply supported on the edges normal to the stiffeners with any other classical boundaries on the other two edges. The plate is modeled using classical plate theory, and each stiffener is modeled as an elementary beam for lateral motion and as an elementary shaft for torsional motion. For simplicity, the effects of warping of the stiffeners and in-plane action are neglected (although they can be included in a more involved analysis). Eccentricity of the stiffeners is included through the moment of inertia of each stiffener. A review of the previously mentioned literature reveals that, in general, the absence of the above-mentioned effects amounts to less than a 5% error in the natural frequencies.

In the section that follows, the governing differential equations for the natural modes of vibration of the stiffened plate are solved using Green's function for the unstiffened rectangular plate. Specialization of the coordinates to those of each stiffener results in a set of coupled homogeneous Fredholm integral equations of the second kind for the unknown displacements and rotations of each stiffener. For the type of stiffened plate considered here Green's functions can be expressed in exact form as a sine series, which allows the exact solution of the integral equations by a standard technique. The result is a set of transcendental characteristic equations for the exact natural frequencies of the stiffened plate. Once the exact natural frequencies have been determined (numerically), the exact natural modes are known in terms of integrals of Green's functions.

For other types of stiffened rectangular plates (e.g., different boundary conditions), the derivation of the set of coupled integral equations for the stiffener displacements and rotations is still valid. However, at this time only approximate forms are available for the Green's functions.

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Also, approximate methods will be necessary to solve the integral equations for the natural frequencies and mode shapes.

An example showing the variation of natural frequencies with stiffener depth ratio, stiffener position, eccentricity, and plate aspect ratio is given for a rectangular plate with a single stiffener.

Finally, the orthogonality relation for the natural modes of the stiffened plate is given. The orthogonality relation is valid for general types of boundary conditions, allowing the forced response of the system, including the effects of damping, to be determined by modal analysis.[16]

Free-Vibration Analysis

Consider a thin, linear elastic, isotropic, rectangular plate of uniform thickness h stiffened by R linear elastic, isotropic beams, each of uniform cross section. The stiffeners are all parallel to an edge of the plate and the edges perpendicular to the stiffeners are simply supported (see Fig. 1). The edges parallel to the stiffeners may have any other classical homogeneous boundary conditions. For simplicity, only the transverse forces $f_i(x,t)$ and twisting moments $m_i(x,t)$ of constraint between the stiffeners and the plate are included. Using the Kirchhoff theory, the equation of motion for the plate is

$$D\nabla^4 w+\rho h\ddot{w}=\sum_{i=1}^{R}[f_i\delta(y-c_i)+m_i\delta'(y-c_i)] \tag{1}$$

where $w(x,y,t)$ is the transverse displacement, D the plate bending stiffness, ρ the mass density of the plate, and ∇^4 the biharmonic operator. The Dirac delta function $\delta(\ )$ and its derivatives satisfy

$$\int_{-\infty}^{\infty} f(y)\delta^{(n)}(y-\eta)\mathrm{d}y=(-1)^n f^{(n)}(\eta) \tag{2}$$

Using the elementary theories for beam bending and twisting,

$$E_iI_iv_i''''+\rho_{s_i}A_i\ddot{v}_i=-f_i \tag{3}$$

$$G_iJ_i\theta_i''-\rho_{s_i}J_i\ddot{\theta}_i=-m_i, \qquad i=1,2,\ldots,R \tag{4}$$

where $v_i(x,t)$ is the transverse displacement of the ith beam, $\theta_i(x,t)$ the rotation, E_iI_i the beam bending stiffness, G_i the shear modulus, J_i the polar moment of inertia, ρ_{s_i} the mass density, and A_i the cross-sectional area. A prime denotes differentiation with respect to x and a dot denotes differentiation with respect to time t. The kinematic constraints for each stiffener are

$$v_i(x,t)=w(x,c_i,t) \tag{5}$$

$$i=1,2,\ldots,R$$

$$\theta_i(x,t)=w,_y(x,c_i,t) \tag{6}$$

For the free-vibration analysis we assume harmonic motion. Thus,

$$w(x,y,t)=W(x,y)\sin\omega t \tag{7a}$$

$$v_i(x,t)=V_i(x)\sin\omega t \tag{7b}$$

$$\theta_i(x,t)=\theta_i(x)\sin\omega t \tag{7c}$$

$$f_i(x,t)=F_i(x)\sin\omega t \tag{7d}$$

$$m_i(x,t)=M_i(x)\sin\omega t \tag{7e}$$

where ω is a frequency. Substituting Eqs. (7) into Eqs. (1), (3), and (4), and using Eqs. (5) and (6) results in the governing equation for the natural modes W;

$$\nabla^4 W-\alpha^4 W=-\sum_{i=1}^{R}\{[\kappa_i W,_{xxxx}(x,c_i)-\mu_i\omega^2 W(x,c_i)]\delta(y-c_i)+[\gamma_i W,_{yxx}(x,c_i)+\sigma_i\omega^2 W,_y(x,c_i)]\delta'(y-c_i)\} \tag{8}$$

where

$$\alpha^4=\rho h\omega^2/D \tag{9a}$$

$$\kappa_i=E_iI_i/D \tag{9b}$$

$$\mu_i=\rho_{s_i}A_i/D \tag{9c}$$

$$\gamma_i=G_iJ_i/D \tag{9d}$$

$$\sigma_i=\rho_{s_i}J_i/D \tag{9e}$$

The governing equation for $W(x,y)$ is solved using Green's function $G(x,y,\xi,\eta;\omega)$ for the vibrating unstiffened rectangular plate. Green's function satisfies the governing equation

$$\nabla^4 G-\alpha^4 G=\delta(x-\xi)\delta(y-\eta) \tag{10}$$

plus the same boundary conditions as the stiffened plate. Thus the solution to Eq. (8) is

$$W(x,y)=-\int_0^a\sum_{i=1}^{R}[\kappa_i W,_{\xi\xi\xi\xi}(\xi,c_i)-\mu_i\omega^2 W(\xi,c_i)]\times G(\xi,c_i,x,y;\omega)\mathrm{d}\xi+\int_0^a\sum_{i=1}^{R}[\gamma_i W,_{\eta\xi\xi}(\xi,c_i)+\sigma_i\omega^2 W,_\eta(\xi,c_i)]G,_\eta(\xi,c_i,x,y;\omega)\mathrm{d}\xi$$
$$0\le x\le a, \quad 0\le y\le b \tag{11}$$

Integration by parts and application of the boundary conditions yields

$$W(x,y)=-\int_0^a\sum_{i=1}^{R}W(\xi,c_i)[\kappa_i G,_{\xi\xi\xi\xi}(\xi,c_i,x,y;\omega)-\mu_i\omega^2 G(\xi,c_i,x,y;\omega)]\mathrm{d}\xi+\int_0^a\sum_{i=1}^{R}W,_\eta(\xi,c_i)[\gamma_i G,_{\eta\xi\xi}(\xi,c_i,x,y;\omega)+\sigma_i\omega^2 G,_\eta(\xi,c_i,x,y;\omega)]\mathrm{d}\xi \tag{12}$$

Equation (12) gives the deflection $W(x,y)$ once $W(x,c_i)$, $i=1,2,\ldots,R$, and ω are determined. Differentiate Eq. (12) with respect to y and let $y\to c_j$, $j=1,2,\ldots,R$ in both Eq. (12) and its differentiated form to obtain the system of coupled homogeneous Fredholm integral equations for the transverse

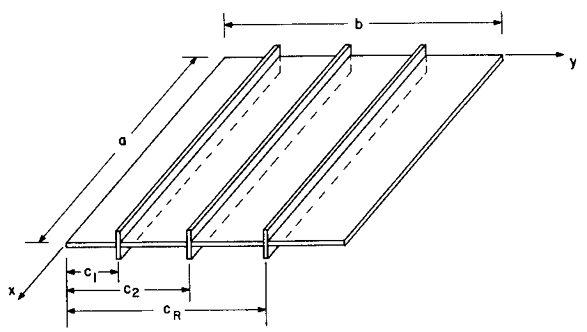

Fig. 1 Rectangular plate with R stiffeners.

deflections $W(x,c_i)=V_i(x)$ and rotations $W,_y(x,c_i)=\theta_i(x)$;

$$V_j(x)=-\int_0^a\sum_{i=1}^R V_i(\xi)K_{ij}(\xi,x;\omega)d\xi$$
$$-\int_0^a\sum_{i=1}^R\theta_i(\xi)L_{ij}(\xi,x;\omega)d\xi \qquad (13)$$
$$\theta_j(x)=-\int_0^a\sum_{i=1}^R V_i(\xi)\hat{K}_{ij}(\xi,x;\omega)d\xi$$
$$-\int_0^a\sum_{i=1}^R\theta_i(\xi)\hat{L}_{ij}(\xi,x;\omega)d\xi,\qquad 0\le x\le a,\quad j=1,2,\ldots,R \qquad (14)$$

where the kernel functions are given by

$$K_{ij}=\kappa_i G,_{\xi\xi\xi\xi}(\xi,c_i,x,c_j;\omega)-\mu_i\omega^2G(\xi,c_i,x,c_j;\omega) \qquad (15a)$$
$$\hat{K}_{ij}=\kappa_i G,_{\xi\xi\xi\xi y}(\xi,c_i,x,c_j;\omega)-\mu_i\omega^2G,_y(\xi,c_i,x,c_j;\omega) \qquad (15b)$$
$$L_{ij}=-\gamma_i G,_{\eta\xi\xi}(\xi,c_i,x,c_j;\omega)-\sigma_i\omega^2G,_\eta(\xi,c_i,x,c_j;\omega) \qquad (15c)$$
$$\hat{L}_{ij}=-\gamma_i G,_{\eta\xi\xi y}(\xi,c_i,x,c_j;\omega)-\sigma_i\omega^2G,_{\eta y}(\xi,c_i,x,c_j;\omega) \qquad (15d)$$

Since the system of integral equations [Eqs. (13) and (14)] is homogeneous, their nontrivial solution results in the natural frequencies ω_n and natural modes $W_n(x,y)$.

The analytical solution of Eqs. (13) and (14) for general boundary conditions is a difficult matter. However, for plates with simply supported edges perpendicular to the stiffeners, Green's function can be expressed in a form that allows a standard technique to be used to arrive at a set of transcendental characteristic equations for the natural frequencies. These characteristic equations represent the exact solution of the integral equations. Since the edges $x=0$ and a are simply supported, the well-known Levy method can be used to obtain Green's function in the form

$$G(x,y,\xi,\eta;\omega)=\sum_{\ell=1}^{\infty}\phi_\ell(y,\eta;\omega)\sin\frac{\ell\pi x}{a}\sin\frac{\ell\pi\xi}{a} \qquad (16)$$

where the functions $\phi_\ell(y,\eta;\omega)$ depend on the boundary conditions at $y=0,b$. Thus

$$K_{ij}=\sum_{\ell=1}^{\infty}B_{ij\ell}(\omega)\sin\frac{\ell\pi\xi}{a}\sin\frac{\ell\pi x}{a} \qquad (17a)$$
$$\hat{K}_{ij}=\sum_{\ell=1}^{\infty}\hat{B}_{ij\ell}(\omega)\sin\frac{\ell\pi\xi}{a}\sin\frac{\ell\pi x}{a} \qquad (17b)$$
$$L_{ij}=\sum_{\ell=1}^{\infty}C_{ij\ell}(\omega)\sin\frac{\ell\pi\xi}{a}\sin\frac{\ell\pi x}{a} \qquad (17c)$$
$$\hat{L}_{ij}=\sum_{\ell=1}^{\infty}\hat{C}_{ij\ell}(\omega)\sin\frac{\ell\pi\xi}{a}\sin\frac{\ell\pi x}{a} \qquad (17d)$$

where

$$B_{ij\ell}(\omega)=[(\ell\pi/a)^4\kappa_i-\mu_i\omega^2]\phi_\ell(c_i,c_j;\omega) \qquad (18a)$$
$$\hat{B}_{ij\ell}(\omega)=[(\ell\pi/a)^4\kappa_i-\mu_i\omega^2]\phi_{\ell,y}(c_i,c_j;\omega) \qquad (18b)$$
$$C_{ij\ell}(\omega)=[(\ell\pi/a)^2\gamma_i-\sigma_i\omega^2]\phi_{\ell,\eta}(c_i,c_j;\omega) \qquad (18c)$$
$$\hat{C}_{ij\ell}(\omega)=[(\ell\pi/a)^2\gamma_i-\sigma_i\omega^2]\phi_{\ell,\eta y}(c_i,c_j;\omega) \qquad (18d)$$

Substituting Eqs. (17) into Eqs. (13) and (14) yields

$$V_j(x)=-\sum_{\ell=1}^{\infty}\sum_{i=1}^{R}[d_{i\ell}B_{ij\ell}(\omega)+e_{i\ell}C_{ij\ell}(\omega)]\sin\frac{\ell\pi x}{a} \qquad (19)$$
$$\theta_j(x)=-\sum_{\ell=1}^{\infty}\sum_{i=1}^{R}[d_{i\ell}\hat{B}_{ij\ell}(\omega)+e_{i\ell}\hat{C}_{ij\ell}(\omega)]\sin\frac{\ell\pi x}{a}$$
$$j=1,2,\ldots,R \qquad (20)$$

where

$$d_{i\ell}=\int_0^a V_i(\xi)\sin\frac{\ell\pi\xi}{a}d\xi \qquad (21)$$
$$e_{i\ell}=\int_0^a \theta_i(\xi)\sin\frac{\ell\pi\xi}{a}d\xi \qquad (22)$$

are constants to be determined. Equations (19) and (20) represent the form of the solution $V_j(x)$ and $\theta_j(x)$ once ω and $d_{i\ell}$ and $e_{i\ell}$ have been determined. Next, operate by

$$\int_0^a\sin\frac{k\pi x}{a}[\text{Eqs. (19) and (20)}]dx,\qquad k=1,2,\ldots \qquad (23)$$

and use the orthogonality of the sine functions to obtain

$$\sum_{i=1}^{R}\left\{\left[\delta_{ij}+\frac{a}{2}B_{ijk}(\omega)\right]d_{ik}+\frac{a}{2}C_{ijk}(\omega)e_{ik}\right\}=0 \qquad (24)$$
$$\sum_{i=1}^{R}\left\{\frac{a}{2}\hat{B}_{ijk}(\omega)d_{ik}+\left[\delta_{ij}+\frac{a}{2}\hat{C}_{ijk}\right]e_{ik}\right\}=0,$$
$$j=1,2,\ldots,R;\quad k=1,2,\ldots \qquad (25)$$

where δ_{ij} is the Kronecker delta. Equations (24) and (25) represent an infinite dimensional set of simultaneous homogeneous transcendental equations for d_{ik} and e_{ik}. A nontrivial solution is assured if, and only if, the determinant of the coefficient matrix is zero. Note that nontrivial solution of Eqs. (24) and (25) is guaranteed if the determinant of the coefficient matrix is zero for any fixed value of k. Thus one need only solve successive $2R\times 2R$ determinantal equations for their roots to obtain the exact natural frequencies. Also note that there are no summations of infinite series associated with the characteristic equations.

Once a natural frequency ω_n and corresponding constants d_{ik_n} and e_{ik_n}, $k=1,2,\ldots$, have been determined, the natural mode $W_n(x,y)$ is given through Eqs. (19), (20), and (12).

Example

As an example consider a rectangular plate simply supported on all sides with a single stiffener of rectangular cross section (see Fig. 2). Thus $R=1$. The stiffener is located at $c_1=c$. For simplicity the plate and stiffener are made of the same material. Thus they have the same modulus of elasticity E, Poisson's ratio ν, and mass density ρ. We will consider two types of stiffeners: one is symmetric with respect to the midplane of the plate and noneccentric, the other is an eccentric stiffener. The symmetric stiffener can be considered to consist of one rib attached to the top of the plate and another rib attached to the bottom. The moment of inertia is

$$I_s=\frac{2dh^3}{3}\left[\left(\frac{f}{h}\right)^3+\frac{3}{2}\left(\frac{f}{h}\right)^2+\frac{3}{4}\left(\frac{f}{h}\right)\right]$$
$$\equiv\frac{2dh^3}{3}g\left(\frac{f}{h}\right) \qquad (26)$$

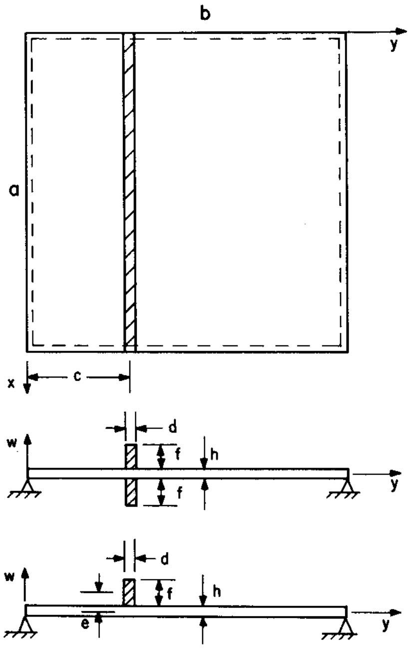

Fig. 2 Simply supported stiffened rectangular plate. Symmetric stiffener, eccentric stiffener.

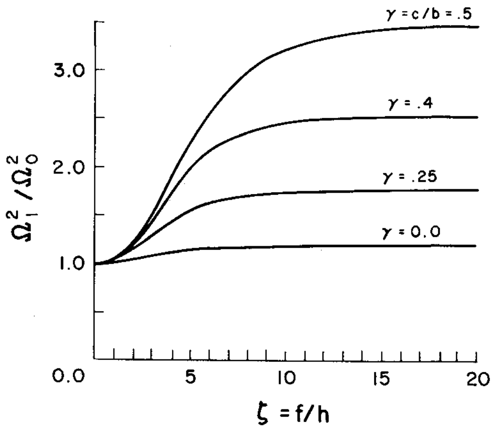

Fig. 3 Fundamental frequency vs stiffener depth ratio. Square plate with symmetric stiffener.

the product of inertia is

$$J_s = 2\left[\frac{dh^3}{3} g\left(\frac{f}{h}\right) + \frac{d^3 f}{12}\right] \tag{27}$$

and the cross-sectional area is

$$A_s = 2fd \tag{28}$$

For the eccentric stiffener, the cross-sectional area is

$$A = fd \tag{29}$$

The moment of inertia (about the midplane of the plate) is

$$I_e = (dh^3/3)g(f/h) = I + e^2 A \tag{30}$$

where $I = df^3/12$ is the moment of inertia of the eccentric stiffener about its own neutral axis, and the eccentricity is

$$e = \tfrac{1}{2}(f+h) \tag{31}$$

The product of inertia is

$$J_e = (dh^3/3)g(f/h) + (d^3 f/12) = J + e^2 A \tag{32}$$

We have assumed that the eccentric stiffener bends about the midplane of the plate.

The functions $\phi_\ell(y,\eta;\omega)$, introduced in Eq. (16), for the unstiffened rectangular plate simply supported on all sides, are[16]

$$\begin{aligned} \phi_\ell(y,\eta,\omega) &= (1/a\alpha^2)\Phi_\ell(y,\eta;\omega), \qquad y \le \eta \\ &= (1/a\alpha^2)\Phi_\ell(\eta,y;\omega), \qquad y \ge \eta \end{aligned} \tag{33a}$$

where

$$\Phi_\ell(y,\eta;\omega) = \frac{-\psi_1(b-\eta)\psi_1(y)}{\psi_1(b)} + \frac{\psi_2(b-\eta)\psi_2(y)}{\psi_2(b)} \tag{33b}$$

$$\psi_1(y) = \frac{\sinh\sqrt{(\ell\pi/a)^2+\alpha^2}\,y}{\sqrt{(\ell\pi/a)^2+\alpha^2}} \tag{33c}$$

and

$$\psi_2(y) = \frac{\sinh\sqrt{(\ell\pi/a)^2-\alpha^2}\,y}{\sqrt{(\ell\pi/a)^2-\alpha^2}}, \qquad \text{if } \alpha < \ell\pi/a \tag{33d}$$

$$= \frac{\sin\sqrt{\alpha^2-(\ell\pi/a)^2}\,y}{\sqrt{\alpha^2-(\ell\pi/a)^2}}, \qquad \text{if } \alpha > \ell\pi/a \tag{33e}$$

$$= y, \qquad \text{if } \alpha = \ell\pi/a \tag{33f}$$

The calculation of the functions $\phi_\ell(y,\eta;\omega)$ given in Eqs. (33) for the unstiffened rectangular plate simply supported on all sides (or simply supported on the edges perpendicular to the stiffener direction with any other classical homogeneous boundary conditions on the other two edges) is accomplished by a straightforward application of the well-known Levy method.

Thus, the characteristic equation for the natural frequencies, using Eqs. (18), (24), (25), and (33), in nondimensional form, is

$$\begin{vmatrix} 1+b_k(\Omega^2) & c_k(\Omega^2) \\ \hat{b}_k(\Omega^2) & 1+\hat{c}_k(\Omega^2) \end{vmatrix} = 0, \quad k = 1,2,\ldots \tag{34}$$

where

$$b_k(\Omega^2) = P_k(\beta,\delta,\epsilon,\zeta,\nu;\Omega^2)\bar{\phi}_k(\gamma,\gamma;\Omega^2;\lambda) \tag{35a}$$

$$\hat{b}_k(\Omega^2) = P_k(\beta,\delta,\epsilon,\zeta,\nu;\Omega^2)\bar{\phi}_{k,y}(\gamma,\gamma;\Omega^2;\lambda) \tag{35b}$$

$$c_k(\Omega^2) = Q_k(\beta,\delta,\epsilon,\zeta,\nu;\Omega^2)\bar{\phi}_{k,\eta}(\gamma,\gamma;\Omega^2;\lambda) \tag{35c}$$

$$\hat{c}_k(\Omega^2) = Q_k(\beta,\delta,\epsilon,\zeta,\nu;\Omega^2)\bar{\phi}_{k,\eta y}(\gamma,\gamma;\Omega^2;\lambda) \tag{35d}$$

$$P_k = \beta\delta[2k^4\pi^4(1-\nu^2)g(\zeta) - \tfrac{1}{2}\zeta\Omega^4] \tag{36a}$$

$$Q_k = \tfrac{1}{2}\beta\delta\epsilon^2[g(\zeta) + \tfrac{1}{4}\zeta\delta^2\epsilon^{-2}][6k^2\pi^2(1-\nu)\epsilon^{-2} - \Omega^4] \tag{36b}$$

$$\delta = d/a \tag{37a}$$

$$\gamma = c/b \tag{37b}$$

$$\epsilon = h/a \tag{37c}$$

$$\zeta = f/h \tag{37d}$$

$$\lambda = b/a \tag{37e}$$

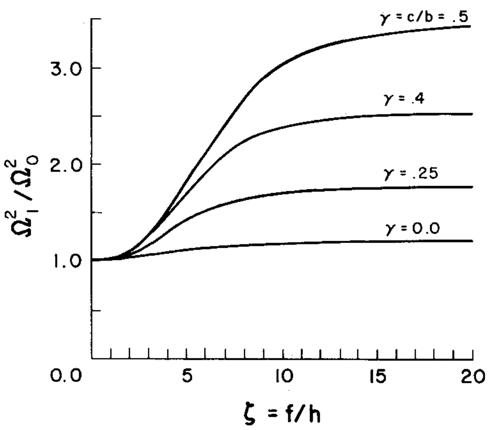

Fig. 4 Fundamental frequency vs stiffener depth ratio. Square plate with eccentric stiffener.

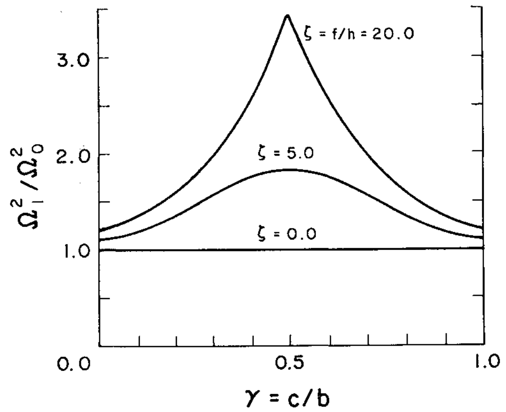

Fig. 6 Fundamental frequency vs stiffener position. Square plate with eccentric stiffener.

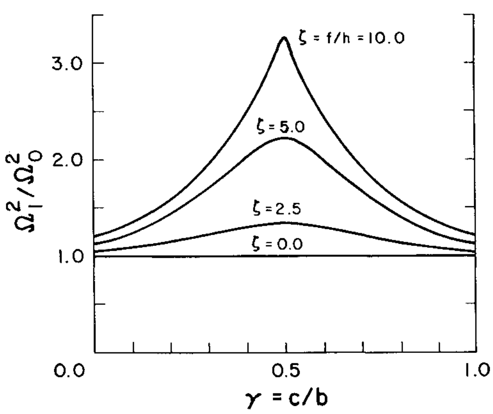

Fig. 5 Fundamental frequency vs stiffener position. Square plate with symmetric stiffener.

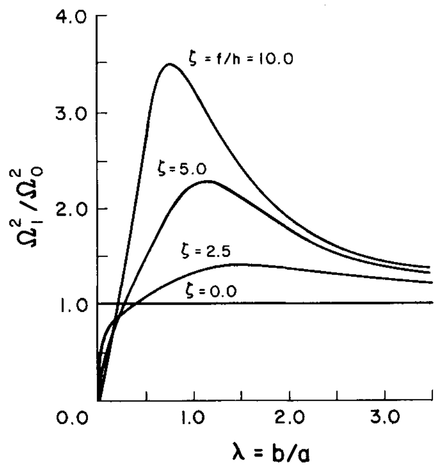

Fig. 7 Fundamental frequency vs plate aspect ratio. Central, symmetric stiffener.

The nondimensional frequency is

$$\Omega^2 = \sqrt{a^4 \rho h / D}\,\omega = a^2 \alpha^2 \tag{38}$$

and $\beta = 1$ for the eccentric stiffener and $\beta = 2$ for the symmetric stiffener. Also,

$$\bar{\phi}_k(\zeta, \zeta; \Omega^2; \lambda) = \frac{1}{\Omega^2}\left\{ -\frac{\bar{\psi}_1(\lambda - \zeta)\bar{\psi}_1(\zeta)}{\bar{\psi}_1(\lambda)} + \frac{\bar{\psi}_2(\lambda - \zeta)\bar{\psi}_2(\zeta)}{\bar{\psi}_2(\lambda)} \right\} \tag{39a}$$

$$\bar{\psi}_1(\zeta) = \frac{\sinh\sqrt{(k\pi)^2 + \Omega^2}\,\zeta}{\sqrt{(k\pi)^2 + \Omega^2}} \tag{39b}$$

$$\bar{\psi}_2(\zeta) = \frac{\sinh\sqrt{(k\pi)^2 - \Omega^2}\,\zeta}{\sqrt{(k\pi)^2 - \Omega^2}} \quad \text{if } \Omega < k\pi \tag{39c}$$

$$= \frac{\sin\sqrt{\Omega^2 - (k\pi)^2}\,\zeta}{\sqrt{\Omega^2 - (k\pi)^2}} \quad \text{if } \Omega > k\pi \tag{39d}$$

$$= \zeta \quad \text{if } \Omega = k\pi \tag{39e}$$

where we have added the parameter λ, denoting the plate aspect ratio, to the argument list of functions $\bar{\phi}_k$. Thus all of the pertinent parameters β, γ, δ, ϵ, ζ, λ, ν, and Ω^2 are explicitly displayed in the characteristic equation. The characteristic equation (34) was solved numerically using a simple root solver (Newton bisection).

Figures 3-8 show the variation of the nondimensional fundamental frequency Ω_1^2 of the stiffened plate normalized by the nondimensional fundamental frequency of the unstiffened simply supported plate $\Omega_0^2 = \pi^2(1 + \lambda^{-2})$. Figures 3 and 4 show the variation of the fundamental frequency for a square plate with stiffener depth ratio $\zeta = f/h$. For small ζ, generally $0 < \zeta < 0.2$, the fundamental frequency decreases slightly (less than 0.01%) due to the added stiffener mass. For higher values of ζ the stiffener raises the fundamental frequency. For $\gamma = c/b = 0.0$, the effect of an edge stiffener (with the additional constraint that lateral displacement of the edge be zero) can be seen. Figures 5 and 6 show the variation of the fundamental frequency for a square plate with stiffener position $\gamma = c/b$. A centrally placed stiffener maximizes the increase in frequency. As a simple check one can see that as the cross-sectional area of the stiffener becomes zero (i.e., $\zeta \to 0$) the fundamental frequency of the stiffened plate approaches that of the simply supported unstiffened plate. Also, as $\zeta \to \infty$, the stiffener acts as a clamped edge at $y = c$ and the fundamental frequency of the stiffened plate approaches that of the corresponding unstiffened rectangular plate with three edges simply supported and the remaining edge clamped. This can be verified with the use of Table 4.13 of Ref. 18. For the special case of a centrally stiffened simply supported rectangular plate, Kirk[8] has derived the exact solution for the characteristic equation of the fundamental frequency by solving the partial differential equations for the vibrating plate. Only the transverse force of constraint is necessary (the twisting moment of constraint being zero for this symmetric mode). Kirk's result has been used to obtain the exact fundamental frequency and, as expected, the results of the present method (exact solution via integral equations) match those of Kirk's exact solution.

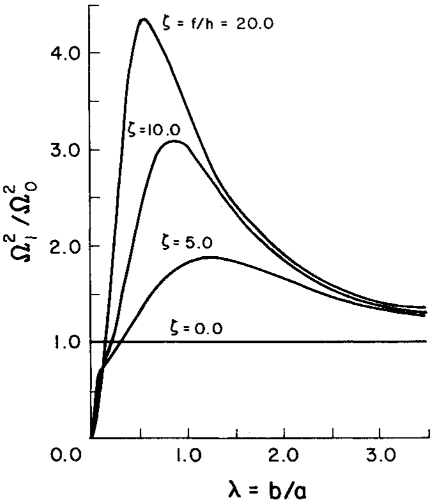

Fig. 8 Fundamental frequency vs plate aspect ratio. Central, eccentric stiffener.

Thus, the accuracy of the present method is further verified. It should be mentioned that Kirk's exact solution is limited to the symmetric modes of centrally stiffened rectangular plates, whereas the present method does not have these restrictions. In addition, Kirk compared the exact solution with a two-term Rayleigh-Ritz procedure and found a less than 3% difference between the results of the exact and approximate methods (see Fig. 5 of Ref. 8).

Figures 7 and 8 show the variation of the fundamental frequency for a centrally stiffened plate with plate aspect ratio $\lambda = b/a$. For $\lambda < 1$, the stiffener is placed in the "long" direction of the rectangular plate; one can see that for small values of λ the stiffener acts as added mass and lowers the fundamental frequency. Table 1 shows the first nine nondimensional frequencies for a square simply supported plate with a central stiffener. The plate is stiffened alternately by symmetric and eccentric stiffeners of equal mass. Thus $\lambda = 1.0$, $\gamma = 0.5$, for the symmetric stiffener $\beta = 2$ and $\zeta = 5.0$, and for the eccentric stiffener $\beta = 1$ and $\zeta = 10.0$. As expected, the eccentric stiffener is more effective at raising the natural frequencies of the plate. Note that all frequencies determined can be considered as exact within the context of the mathematical model chosen. The ability to compute the higher frequencies in an exact manner represents an advantage of the present method over approximate methods such as finite elements, finite differences, Lagrange multipliers, Rayleigh-Ritz, etc. For all examples $\delta = \epsilon = 0.01$ and $\nu = 0.3$. For an eccentric stiffener, the eccentricity ratio can be computed using

$$e/h = \tfrac{1}{2}\delta(1+\zeta) \tag{40}$$

Orthogonality Relation

The orthogonality relation for the natural modes of the stiffened plate can be derived by a classical procedure[16] using Eqs. (1-7) with $\omega = \omega_n$. The orthogonality relation is

$$\int_0^b \int_0^a \rho h W_m(x,y) W_n(x,y) \mathrm{d}x\mathrm{d}y + \sum_{i=1}^{R} \left\{ \int_0^a \rho_s [A_i V_{i_m}(x) V_{i_n}(x) + J_i \theta_{i_m}(x) \theta_{i_n}(x)] \mathrm{d}x \right\} = d_n \delta_{mn}, \qquad m,n = 1,2,\ldots \tag{41}$$

where d_n is the normalization constant. The forced response of the stiffened plate to arbitrary excitation including the effects of damping can now be determined by modal analysis.[16]

Table 1 Nondimensional frequencies Ω_n^2/Ω_0^2 for a square plate with a central stiffener

n	Symmetric ($f/h = 5$)	Eccentric ($f/h = 10$)
1	2.215	3.012
2	2.986	3.271
3	4.594	4.723
4	4.615	4.742
5	5.089	6.390
6	7.010	7.073
7	7.047	7.087
8	9.194	9.820
9	10.295	10.455

Conclusion

A new method for the free-vibration analysis of rectangular plates with stiffeners parallel to an edge of the plate has been demonstrated. The effects of the transverse forces and twisting moments between the plate and stiffeners are accounted for. The method is based on the use of Green's function for the vibrating unstiffened plate and results in a set of coupled homogeneous Fredholm integral equations of the second kind for the stiffener transverse displacements and rotations. Solution of the integral equations is exact when the edges of the plate perpendicular to the stiffeners are simply supported. The result is a set of characteristic equations for the exact natural frequencies of the stiffened plate. In nondimensional form, all of the pertinent parameters of the problem appear explicitly in these transcendental equations for the natural frequencies. Thus, one possible extension of the method is its use in design as a tool that allows mathematical optimization (e.g., derivation of optimization equations through differentiation with respect to a parameter, or by minimizing a cost functional) in addition to parameter studies.

Similar methods can be used for the linear buckling analysis of stiffened rectangular plates although the effects of twisting of the stiffeners would need to be handled in a different manner. Thus no direct comparison of fundamental frequencies and plate buckling loads can be made in the examples presented.

The characteristic equations for the exact natural frequencies are solved numerically. A short computer program was written that easily can be run on a microprocessor or portable computer. Typical CPU times on the University of Illinois at Urbana-Champaign CYBER 175 computer necessary to determine the fundamental frequency of the stiffened plate to eight significant figures for a given set of parameters were on the order of 0.05 s. The number of significant figures to which a frequency is calculated does not significantly affect the amount of CPU time. A table and figures show some typical results for a simply supported rectangular plate with a single stiffener of rectangular cross section.

A particular advantage of the method is that the exact natural modes can be computed quite easily, and, along with the orthogonality relation for the natural modes, allow the forced response to arbitrary excitation to be determined by modal analysis. This increases the potential of the method further for use in design, optimization, and control of stiffened plates.

The inclusion of warping of the stiffeners and in-plane effects is possible. However, this will further restrict the types of boundary conditions for which exact results can be obtained. In general, the integral equations obtained for stiffener transverse deflections, rotations, etc., will be exact for classical homogeneous boundary conditions. However, for plates that do not have simply supported edges perpendicular to the stiffeners, the determination of Green's functions and, hence, the kernel functions, will be approximate (e.g., by Galerkin's method), the alternate, approximate methods of solving the integral equations will be necessary. Still, accurate results at reasonable cost should be forthcoming.

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